From dictionaries to schemes: why order matters

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Acknowledgments

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- Cristina Bertone (University of Turin),
- Francesca Cioffi (University of Naples),
- Werner M. Seiler (University of Kassel).

Lexicographic order: applied . . .

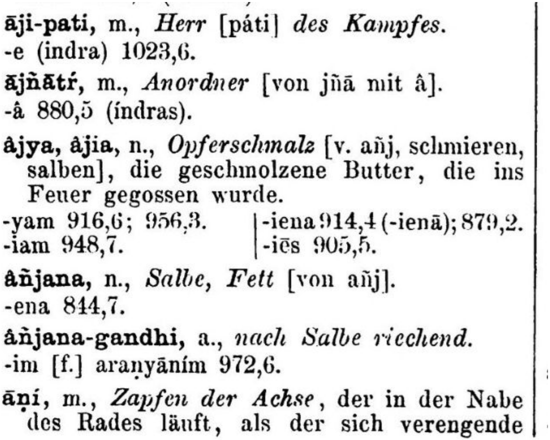
āji-pati, m., *Herr* [páti] *des Kampfes.*
-e (indra) 1023,6.
ājñātŕ, m., *Anordner* [von jñā mit â].
-â 880,5 (índras).
âjya, âjia, n., *Opferschmalz* [v. añj, schmieren, salben], die geschmolzene Butter, die ins Feuer gegossen wurde.
-yam 916,6; 956,3. | -iena 914,4 (-ienā); 879,2.
-iam 948,7. | -iēs 905,5.
âñjana, n., *Salbe, Fett* [von añj].
-ena 844,7.
âñjana-gandhi, a., *nach Salbe riechend.*
-im [f.] araṇyāním 972,6.
āṇí, m., *Zapfen der Achse*, der in der Nabe des Rades läuft, als der sich verengende

Figure: Extract from a dictionary of the Rig Veda, 1873 (in German).

. . . and somewhat more theoretic

Let's ignore hyphens and diacritics and work in the alphabet $a, b, c, \ldots, z$.

ajipati
ajnatr
ajya
anjana
anjanagandhi
ani

. . . and somewhat more theoretic

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$$\begin{array}{lcl} \text{ajipati} & & \text{aaaadghijnnn} \\ \text{ajnatr} & & \text{aaajnn} \\ \text{ajya} & & \text{aaiijpt} \\ \text{anjana} & \rightsquigarrow & \text{aajnrt} \\ \text{anjanagandhi} & & \text{aajy} \\ \text{ani} & & \text{ain} \end{array}$$

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\text{ajipati} & & \text{aaaadghijnnn} & & a^4dghijn^3 \\
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\text{anjana} & \rightsquigarrow & \text{aajnrt} & \rightsquigarrow & a^2jnrt \\
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\end{array}$$

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ajnatr		aaajnn		a^3jn^2		$x_1^3x_{10}x_{14}^2$
ajya		aaiijpt		a^2i^2jpt		$x_1^2x_9^2x_{10}x_{16}x_{20}$
anjana	$\rightsquigarrow$	aajnrt	$\rightsquigarrow$	a^2jnrt	$\rightsquigarrow$	$x_1^2x_{10}x_{14}x_{18}x_{20}$
anjanagandhi		aajy		a^2jy		$x_1^2x_{10}x_{25}$
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$$

- *Monomials* in commuting *variables* $x_1, \ldots, x_{26}$ ordered *lexicographically*.

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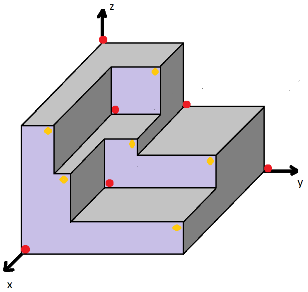
ani		ain		ain		$x_1x_9x_{14}$

- *Monomials* in commuting *variables* $x_1, \dots, x_{26}$ ordered *lexicographically*.
- Let $\mathbb{K}$ be a field. *Monomial ideal*: $\mathbb{K}$-linear combinations of monomials divisible by finitely many given ones.



Enter the protagonists

Figure: David Hilbert.

Figure: Hermann Graßmann.

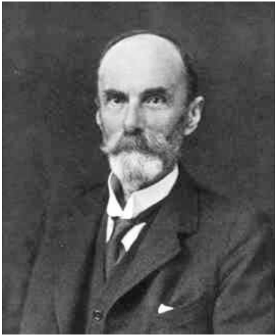


Figure: Francis Sowerby Macaulay.

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Theorem (Hartshorne 1966)

For a given Hilbert polynomial, the Hilbert scheme is connected.

Our contributions

Our work has both computational and theoretical flavors. We employ *marked bases* (intuitively: term-order free Gröbner bases) and their analogues in quotient rings for studying Hilbert schemes.

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Theorem (BCOS 2026)

Marked bases give effective methods for detecting and constructing Cohen–Macaulay, Gorenstein, and complete intersection ideals.

What's next?

$x_1^4 x_4 x_7 x_8 x_9 x_{10} x_{14}^3$
$x_1^3 x_{10} x_{14}^2$
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Thank you